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Correlation functions of eigenvalues of multi-matrix models, and the limit of a time-dependent matrix

B Eynard†

Department of Mathematical Sciences, University of Durham, Science Laboratories, South Road, Durham DH1 3HP, UK

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Abstract. The universality of correlation functions of eigenvalues of large random matrices has been observed in various physical systems, and proved in some particular cases, as the Hermitian one-matrix model with polynomial potential. Here, we consider the more difficult case of a unidimensional chain of Hermitian matrices with first-neighbour couplings and polynomial potentials.

An asymptotic expression of the orthogonal polynomials and a generalization of the Darboux–Christoffel theorem allow us to find new results for the correlations of eigenvalues of different matrices of the chain.

Eventually, we consider the limit of the infinite chain of matrices, which can be interpreted as a time-dependent Hermitian one-matrix model, and give the correlation functions of eigenvalues at different times.

Random matrix theory (RMT) was introduced by Wigner [1] to describe statistically the intricated structure of energy levels of heavy nuclei. More recently, the spectra of many quantum systems (presenting chaos or disorder) in condensed matter physics have been observed [2] to be well described by RMT. The properties of such quantum systems are indeed modelled by the eigenvalues and eigenstates of some operators or matrices (Hamiltonian, transfer matrix, scattering matrix), see [3, 4] for a review of RMT in quantum physics.

The main reason RMT provides such a powerful tool to study quantum systems is its universality. As the distribution of a large number of independent random variables converges towards a Gaussian law (central limit theorem), the distribution of eigenvalues of large random matrices seems to converge towards one of the three Gaussian laws (GOE, GUE, GSE) depending on the symmetries of the matrices: orthogonal (time reversibility), unitary (time reversibility broken) or symplectic (half-integer spin dependence).

For instance, it has been observed or proved for a wide number of models that the two-point connected correlation function does not depend on the details of the system [2, 5, 6] in the short-distance regime (of the order of the mean spacing), while the density of levels is very dependent on the specific details of the system. In addition, it has been observed [5, 7, 8], at least in the case of Hermitian matrices, that the correlation functions should present some universality in the long-distance regime as well, once the short-distance oscillations have been smoothed out.

† E-mail address: bertrand.eynard@durham.ac.uk

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At least for Hermitian matrices, those conjectures [9] have been proved in a few special cases, such as in the one-Hermitian-matrix model with polynomial potential [1, 5, 6, 10], the one-Hermitian-matrix model in an external potential [11], and the two-Hermitian-matrix model [12].

The analysis of [5, 12] was based on the method of orthogonal polynomials. The correlation functions are expressed in terms of *kernels*, depending on two variables, which are sums of polynomials. Those results are exact and have been known for a long time [13]. The problem was to derive an asymptotic expansion of the kernels in the large N limit (N is the size of the matrices), which is a hard task since the kernels involve the sum of N polynomials with degrees running from 0 to $N-1$. In the one-matrix model case [5], the Darboux–Christoffel theorem allowed one to rewrite the kernel with only two polynomials of degree N and $N-1$. An ansatz for the asymptotic expression of the orthogonal polynomials was then used to evaluate the universal correlation functions in the short-range regime.

This ansatz was proved in [6, 12] and generalized to the two-matrix case, and it was claimed in [12] that this method could probably be extended to a chain of p random Hermitian matrices $M_1, \ldots, M_p$, where each matrix M_k is coupled linearly to the following one M_{k+1}.

In particular, when the number of matrices of the chain becomes infinite and the coupling is chosen appropriately, this model can be viewed as a time-dependent random matrix. The coupling between neighbouring matrices of the chain is then a kinetic term of the form $(\mathrm{d}M/\mathrm{d}t)^2$.

The aim of this paper is thus to generalize the analysis of [5, 12] to the chain of Hermitian matrices. The paper is organized as follows. Section 1 concerns the discrete chain and section 2 the continuous-time limit. In section 1, we first present the matrix model, recall the orthogonal polynomial method, and then relate the correlation functions to the orthogonal polynomials via the kernels and generalize the Darboux–Christoffel theorem in order to rewrite these kernels as a sum of a finite number of terms. A WKB approximation of the orthogonal polynomials allows one to find asymptotic expressions of the kernels, and thus to find the correlation functions in the large N limit. We then conclude by examining the universal properties of these correlations and the possible generalization of the method.

1. The chain of matrices

Let us first present the model and introduce the notation coherent with that of [12].

Consider a linear chain of $p+1$ random Hermitian $N \times N$ matrices M_i $(0 \leqslant i \leqslant p)$, with a probability law

$$\mathcal{P}(M_i) = \frac{1}{Z} \prod_{i=0}^{p} \exp[-N \operatorname{tr} V_i(M_i)] \prod_{i=0}^{p-1} \exp[-Nc \operatorname{tr}(M_i - M_{i+1})^2] \tag{1.1}$$

where the V_i are polynomial potentials, c is the coupling constant between nearest neighbours, and Z is the partition function. (In section 2, we will consider the continuum limit of this model: the index i will become a continuous variable, the time $t = i\epsilon$, and with $2c = 1/\epsilon$, the quadratic term $\sum_i c \operatorname{tr}(M_i - M_{i+1})^2$ will become a kinetic term $\int \frac{1}{2}\dot{M}^2(t)\,\mathrm{d}t$.)

The Harish–Chandra–Itzykson–Zuber formula [14] allows us to integrate out the angular variables (the unitary group), and leaves us with the joint probability for the eigenvalues

(let us note $\lambda_{i;j}$ $(0 \leqslant j \leqslant N-1)$ the jth eigenvalue of the matrix M_i):

$$\rho[\lambda_{i;j}] = \frac{1}{Z}\Delta(\lambda_1)\Delta(\lambda_p)\prod_{i=0}^{p}\exp\left[-N\sum_j V_i(\lambda_{i;j})\right]\prod_{i=0}^{p-1}\det_{k,l}|\exp[-Nc(\lambda_{i;k}-\lambda_{i+1;l})^2]| \tag{1.2}$$

where Δ are the Vandermonde determinants:

$$\Delta(\lambda_i) = \prod_{k<l}(\lambda_{i;k}-\lambda_{i;l}).$$

We would now like to compute the conditional probabilities of some subset I of these $N\times(p+1)$ eigenvalues. We thus have to integrate (1.2) over all the eigenvalues which do not belong to I. For instance, the density of the eigenvalues of M_i is

$$\rho_i(\lambda_{i;1}) = \int \rho[\lambda] \prod_{(j,k)\neq(i,1)} \mathrm{d}\lambda_{j;k}$$

the correlation function of two eigenvalues of M_i is

$$\rho_{ii}(\lambda_{i;1},\lambda_{i;2}) = \int \rho[\lambda] \prod_{(j,k)\neq(i,1),(i,2)} \mathrm{d}\lambda_{j;k}$$

and the correlation function of two eigenvalues of two matrices M_i and M_j is

$$\rho_{ij}(\lambda_{i;1},\lambda_{j;1}) = \int \rho[\lambda] \prod_{(l,k)\neq(i,1),(j,1)} \mathrm{d}\lambda_{l;k}.$$

As in the one-matrix case [5, 12], all these densities and correlation functions can be calculated by the orthogonal polynomials method [15]. Let us recall this method [4].

1.1. Orthogonal polynomials

Consider two families of polynomials $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$, of degree n, beginning with the same leading term, and which obey the orthogonality relation

$$\int \mathrm{d}\lambda_0 \ldots \mathrm{d}\lambda_p \exp\left[-N\sum_i V_i(\lambda_i)\right]\exp\left[-Nc\sum_i(\lambda_i-\lambda_{i+1})^2\right]\mathcal{P}_n(\lambda_0)\tilde{\mathcal{P}}_m(\lambda_p) = \delta_{n,m}. \tag{1.3}$$

Such polynomials always exist when the V_i's are polynomials, except in some degenerate cases. We then define the wavefunctions ψ_n and $\tilde{\psi}_n$ by

$$\psi_n(\lambda_0) = \mathcal{P}_n(\lambda_0)\exp[-N\tfrac{1}{2}V_0(\lambda_0)]$$
$$\tilde{\psi}_n(\lambda_p) = \tilde{\mathcal{P}}_n(\lambda_p)\exp[-N\tfrac{1}{2}V_p(\lambda_p)]$$

(note that the normalizations differ from [12]). With the help of the orthogonality relation (1.3), we can define two families of Hilbert spaces E_i, $\tilde{E}_i$, and the orthogonal functions in each of them:

$$\psi_{0,n}(\lambda_0) = \psi_n(\lambda_0)$$
$$\psi_{i;n}(\lambda_i) = \int \mathrm{d}\lambda_0 \ldots \mathrm{d}\lambda_{i-1}\,\psi_n(\lambda_0)\exp[-Nc((\lambda_0-\lambda_1)^2+\cdots+(\lambda_{i-1}-\lambda_i)^2)]$$
$$\times\exp\left[-N\left(\frac{1}{2}V_0(\lambda_0)+V_1(\lambda_1)+\cdots+V_{i-1}(\lambda_{i-1})+\frac{1}{2}V_i(\lambda_i)\right)\right] \tag{1.4}$$

$$\tilde{\psi}_{p;n}(\lambda_p) = \tilde{\psi}_n(\lambda_p)$$
$$\tilde{\psi}_{i;n}(\lambda_i) = \int \mathrm{d}\lambda_{i+1} \dots \mathrm{d}\lambda_p \, \tilde{\psi}_n(\lambda_p) \exp[-Nc((\lambda_i - \lambda_{i+1})^2 + \dots + (\lambda_{p-1} - \lambda_p)^2)] \times \exp\left[-N\left(\frac{1}{2}V_i(\lambda_i) + V_{i+1}(\lambda_{i+1}) + \dots + V_{p-1}(\lambda_{p-1}) + \frac{1}{2}V_p(\lambda_p)\right)\right]. \tag{1.5}$$

We shall denote them by the convenient Dirac notation:

$$\langle n|_i = \psi_{i;n} \qquad |n\rangle_i = \tilde{\psi}_{i;n}. \tag{1.6}$$

In the space E_i, we have the orthogonality relation

$$\int \mathrm{d}\lambda_i \, \psi_{i;n}(\lambda_i)\tilde{\psi}_{i;m}(\lambda_i) = \langle n|m\rangle_i = \delta_{n,m}.$$

In each of these spaces, we can define the usual operators (acting on the right-hand side, the ket):

- $\hat{\lambda}_i$, the operator which multiplies $\tilde{\psi}_i(\lambda_i)$ by λ_i;
- $\hat{P}_i = (1/N)\partial/\partial\lambda_i$ which differentiates $\tilde{\psi}_i$ with respect to λ_i.

These operators are defined only in the Hilbert space E_i. However, all the E_i are isomorphic, and an operator $\hat{O}$ initially defined in E_i can also be defined in any E_j by its matrix elements:

$$\langle n|\hat{O}|m\rangle_j \stackrel{\text{def}}{=} \langle n|\hat{O}|m\rangle_i = \int \mathrm{d}\lambda_i \, \psi_{i;n}(\lambda_i) O \tilde{\psi}_{i;m}(\lambda_i).$$

Henceforth, we will drop the index i for the bras and kets.

1.2. Equations of motion

From the former definitions we immediately obtain the equations of motion:

$$\hat{P}_i = 2c(\hat{\lambda}_{i+1} - \hat{\lambda}_i) - \tfrac{1}{2}V_i'(\hat{\lambda}_i) \tag{1.7}$$
$$\hat{P}_0 = 2c(\hat{\lambda}_1 - \hat{\lambda}_0) - \tfrac{1}{2}V_0'(\hat{\lambda}_0) \qquad \hat{P}_p = 2c(\hat{\lambda}_p - \hat{\lambda}_{p-1}) + \tfrac{1}{2}V_p'(\hat{\lambda}_p) \tag{1.8}$$

and with an integration by parts

$$\hat{\lambda}_{i-1} + \hat{\lambda}_{i+1} - 2\hat{\lambda}_i = \frac{1}{2c}V_i'(\hat{\lambda}_i). \tag{1.9}$$

Let us now introduce a more convenient notation. Since we began with polynomials $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$, we know how the multiplications or derivations by λ_0 or λ_p will act: multiplication by λ_p raises the degree of $\tilde{\mathcal{P}}_n$ by 1, and $\lambda_p\tilde{\mathcal{P}}_n(\lambda_p)$ can be decomposed onto the base of the $\tilde{\mathcal{P}}_{n-k}$ with $k \geqslant -1$:

$$\lambda_p\tilde{\mathcal{P}}_n(\lambda_p) = \alpha(n)\tilde{\mathcal{P}}_{n+1}(\lambda_p) + \sum_{k\geqslant 0}\tilde{\alpha}_k(n)\tilde{\mathcal{P}}_{n-k}(\lambda_p)$$

(where $\alpha(n)$ is the ratio of the leading coefficients of $\tilde{\mathcal{P}}_n$ and $\tilde{\mathcal{P}}_{n+1}$, and the $\tilde{\alpha}_k(n)$ are coefficients to be determined later).

Let us write this in operatorial notation. For this purpose, we introduce the shift operator $\hat{x}$, which decreases the level (annihilation operator), and the level operator $\hat{n}$:

$$\hat{x}|n\rangle = |n-1\rangle \qquad \langle n|\hat{x} = \langle n+1| \qquad \hat{n}|n\rangle = n|n\rangle.$$

Although $\hat{x}$ is not invertible, we shall abusively write $\hat{x}^\dagger = 1/\hat{x}$, for it will make no difference when we go to the large N limit, and it will considerably simplify the notation†.

We can then write

$$\hat{\lambda}_0^\dagger = \frac{1}{\hat{x}}\alpha(\hat{n}) + \sum_{k\geqslant 0}\hat{x}^k\alpha_k(\hat{n}) \qquad \hat{\lambda}_p = \frac{1}{\hat{x}}\alpha(\hat{n}) + \sum_{k\geqslant 0}\hat{x}^k\tilde{\alpha}_k(\hat{n}). \tag{1.10}$$

Remember that $\hat{\lambda}_0$ acts on the ket $|n\rangle$, i.e. on the polynomial $\tilde{\mathcal{P}}_n$, its adjoint acts on the bra $\mathcal{P}_n$. Note also that the first term, $\alpha(n)$, is the same for both λ_0 and λ_p because we have chosen the polynomials $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$ with the same leading coefficient.

Similarly, noting that the operator $\hat{P} = (1/N)\mathrm{d}/\mathrm{d}\lambda_p$ decreases the degree of the polynomial $\tilde{\mathcal{P}}_n(\lambda_p)$ by 1,

$$\frac{\mathrm{d}}{\mathrm{d}\lambda_p}\tilde{\mathcal{P}}_n(\lambda_p) = \frac{1}{\alpha(n-1)}n\tilde{\mathcal{P}}_{n-1} + \cdots$$

we can express the operators $\hat{P}_0$ and $\hat{P}_p$ in power series of $\hat{x}$:

$$\hat{P}_0^\dagger + \frac{1}{2}V_0'(\hat{\lambda}_0^\dagger) = \frac{1}{\alpha(\hat{n})}\hat{x}\frac{\hat{n}}{N} + \mathrm{O}(\hat{x}^2) \tag{1.11}$$

$$\hat{P}_p + \frac{1}{2}V_p'(\hat{\lambda}_p) = \frac{1}{\alpha(\hat{n})}\hat{x}\frac{\hat{n}}{N} + \mathrm{O}(\hat{x}^2). \tag{1.12}$$

We might as well write any of the operators $\hat{\lambda}_i$ with such a notation,

$$\hat{\lambda}_i = \sum_k \alpha_{i;k}(\hat{n})\hat{x}^k$$

but let us first go to the large N limit.

1.3. Large N limit

In the classical limit $N \to \infty$, all these operators become numbers. Indeed, the commutators $[\hat{P}, \hat{\lambda}] = 1/N$ and $[\hat{x}, \hat{n}/N] = \hat{x}/N$ are proportional to $1/N$ which thus plays the role of $\hbar$‡. We then write

$$\lambda_i(x) = \sum_k \alpha_{i,k}x^k \qquad \text{with} \qquad -\prod_{j=i+1}^{p}\deg V_j' \leqslant k \leqslant \prod_{j=0}^{i-1}\deg V_j'. \tag{1.13}$$

The bounds on k are easily derived from the equations of motion and boundary conditions. We also consider the limit where n is large and close to N, so that to leading order the $\alpha_{i,k}$ no longer depend on n, rather they are just coefficients.

In addition, there exists a remarkable relation (the proof from the canonical commutation relations is not difficult but of no interest for what follows):

$$1 = c\sum_k k(\alpha_{i+1,k} - \alpha_{i-1,k})\alpha_{i,-k} \qquad 0 \leqslant i \leqslant N. \tag{1.14}$$

† We have $\hat{x}|0\rangle = 0$, and $\hat{x}$ is not invertible only on one state $|0\rangle$. This will have no consequence because all the physics takes place at the 'Fermi level' $|N\rangle$. One way to give a rigorous meaning to $\hat{x}^{-1}$ could be to define negative states $|-1\rangle$, provided that all the $\alpha_k(-1)$ vanish, which is true.

‡ Actually, this is true only if the support of the density is connected, i.e. we assume we have a one-cut solution; for a k-cut solution, we would need to consider the operators as $k \times k$ matrices. For example, for a symmetric double well, one needs to distinguish between even and odd values of n, which introduces two sets of coefficients $\alpha_k(2n)$ and $\alpha_k(2n+1)$.

Let us rewrite in the classical limit the equations of motion (1.9) and the boundary conditions (1.11) and (1.12) previously written for operators. We have the following system of equations:

$$\lambda_{i-1}+\lambda_{i+1}-2\lambda_i=\frac{1}{2c}V_i'(\lambda_i)\qquad 1\leqslant i\leqslant p-1 \tag{1.15}$$

with the boundary conditions

$$V_0'(\lambda_0)-2c(\lambda_1-\lambda_0)=\frac{1}{\alpha x}+\mathrm{O}(1/x^2) \tag{1.16}$$

$$V_p'(\lambda_p)-2c(\lambda_{p-1}-\lambda_p)=\frac{x}{\alpha}+\mathrm{O}(x^2). \tag{1.17}$$

One can verify that we have exactly as many equations as unknowns. If we were able to solve this system of algebraic equations and determine all the $\alpha_{i,k}$, we could define $p+1$ functions $\lambda_i(x)$, of an auxiliary variable x. We will see later the important role they play.

1.4. WKB approximation

One can find (by a simple generalization of [12], i.e. by performing a kind of saddle-point method for matrix integrals on the explicit expressions given in appendix B) some asymptotic expressions of the $\psi_{i;n}$ in the limit N large and $|n-N|\sim\mathrm{O}(1)$:

$$\begin{aligned}
\psi_{0;n}(\lambda_0)&\sim\sum_{x/\lambda_0(x)=\lambda_0}\left(\frac{\pi}{Nc}\right)^{-p/4}\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda_0'(x)}x^{n-N}\exp\left[-2Nc\int^x(\lambda_1-\lambda_0)\lambda_0'\right]\\
&\quad\times\exp\left[\frac{N}{2}V_0(\lambda_0)\right]\\
\psi_{i;n}(\lambda_i)&\sim\sum_x\left(\frac{\pi}{Nc}\right)^{i/2-p/4}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\lambda_i'(x)}}x^{n-N}\exp\left[-Nc\int^x(\lambda_{i+1}-\lambda_{i-1})\lambda_i'\right]\\
\psi_{p;n}(\lambda_p)&\sim\sum_x\left(\frac{\pi}{Nc}\right)^{p/4}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\lambda_p'(x)}}x^{n-N}\exp\left[-2Nc\int^x(\lambda_p-\lambda_{p-1})\lambda_p'\right]\\
&\quad\times\exp\left[-\frac{N}{2}V_p(\lambda_p)\right]\\
\tilde\psi_{0;n}(\lambda_0)&\sim\sum_x\left(\frac{\pi}{Nc}\right)^{p/4}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{-\lambda_0'(x)}}x^{N-n-1}\exp\left[2Nc\int^x(\lambda_1-\lambda_0)\lambda_0'\right]\\
&\quad\times\exp\left[-\frac{N}{2}V_0(\lambda_0)\right]\\
\tilde\psi_{i;n}(\lambda_i)&\sim\sum_x\left(\frac{\pi}{Nc}\right)^{p/4-i/2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{-\lambda_i'(x)}}x^{N-n-1}\exp\left[Nc\int^x(\lambda_{i+1}-\lambda_{i-1})\lambda_i'\right]\\
\tilde\psi_{p;n}(\lambda_p)&\sim\sum_x\left(\frac{\pi}{Nc}\right)^{-p/4}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{-\lambda_p'(x)}}x^{N-n-1}\exp\left[2Nc\int^x(\lambda_p-\lambda_{p-1})\lambda_p'\right]\\
&\quad\times\exp\left[\frac{N}{2}V_p(\lambda_p)\right].
\end{aligned} \tag{1.18}$$

We shall not prove those asymptotic expressions, but just give some intuitive explanations.

• First, observe that to leading order they all have the form

$$\psi_i \sim \exp\left[-N \int^{\lambda_i} P_i \, \mathrm{d}\lambda_i \right] \qquad \text{and} \qquad \tilde{\psi}_i \sim \exp\left[N \int^{\lambda_i} P_i \, \mathrm{d}\lambda_i \right]$$

which is simply the solution of the differential equation $\hat{P}_i|n\rangle = (1/N)(\mathrm{d}|n\rangle/\mathrm{d}\lambda_i)$ in the large N limit. The lower bound of the integral, which has not been written here for simplicity, depends on i; it is determined by the condition that $\int^{\lambda_i} V_i'(\mu_i)\,\mathrm{d}\mu_i = V_i(\lambda_i)$.

• The x^{n-N} term comes from the definition of $\hat{x}$:

$$\langle N|\hat{x}^{n-N} = \langle n|.$$

• Moreover, observe that the approximation for ψ_{i+1} can be derived from ψ_i by steepest descent in (1.4), and the expressions for the $\tilde{\psi}$'s can be derived from the ψ's by $x \to 1/x$ and $i \to p - i$.

• Finally, the normalization constants and $1/\sqrt{2\pi\lambda_i'(x)}$ are just what is needed to satisfy the normalization condition

$$\langle n|m\rangle = \delta_{nm}.$$

Note that this is nothing else than the WKB approximation.

Remember that, in quantum mechanics, the wavefunction of a particle outside a potential well decreases exponentially, while inside the well it is a stationary wave, i.e. a superposition of two opposite progressing waves. This is also what we have here.

• The sum over x means that you have to consider the values of x, solutions of $\lambda_i(x) = \lambda_i$ which have this property. When λ_i belongs to $[a_i, b_i]$ (the support of the density of eigenvalues of the ith matrix), the equation $\lambda_i(x) = \lambda_i$ has no real solution, it has only pairs of complex conjugate solutions, which give the stationary wave. The sum of the two complex solutions will give rise to some real expression for $\psi_{i;n}$, involving cosine and sine functions instead of exponentials (cf [5, 12]). When λ_i is outside $[a_i, b_i]$, you have to keep only the solution which decreases exponentially at infinity.

Henceforth, we will consider only the first case, i.e. $\lambda_i \in [a_i, b_i]$.

1.5. Kernels

Remember that we have introduced the orthogonal polynomials in order to integrate the joint density (1.2) over a subset I of the $N \times (p+1)$ variables [4, 16]. For this purpose, let us as usual [4] rewrite the Vandermonde determinants:

$$\Delta(\lambda_0) = \prod_{k<l} (\lambda_{0;k} - \lambda_{0;l}) = \det_{k,l} |(\lambda_{0;l})^{k-1}|.$$

Since linear combinations of columns preserve the determinant, we can rewrite

$$\Delta(\lambda_0) = \mathrm{cte} \det_{k,l} |\mathcal{P}_{k-1}(\lambda_{0;l})|$$

$$\Delta(\lambda_p) = \mathrm{cte} \det_{k,l} |\tilde{\mathcal{P}}_{k-1}(\lambda_{p;l})|.$$

The cte is a normalization which comes from the fact that the polynomials $\mathcal{P}_k$ and $\tilde{\mathcal{P}}_k$ are not monic (in fact, cte $= \prod_{n=0}^{N-1} \alpha(n)^{N-1-n}$). Any partial integration of (1.2) can thus be written as an integral over the $\psi_{i;n}$ and $\tilde{\psi}_{j;m}$. Since they are orthogonal, the integration is easily performed, and the final result can be written in terms of $2(p+1)^2$ kernels defined by

$$K_{i,j}(\lambda_i, \lambda_j) = \frac{1}{N} \sum_{n=0}^{N-1} \psi_{i;n}(\lambda_i) \tilde{\psi}_{j;n}(\lambda_j) \tag{1.19}$$

and

$$E_{i,j}(\lambda_i,\lambda_j)=\begin{cases}0 & \text{if } i\geqslant j\\ \exp\left\{-\dfrac{N}{2}[2c(\lambda_i-\lambda_{i+1})^2+V_i(\lambda_i)+V_{i+1}(\lambda_{i+1})]\right\} & \text{if } i=j-1\\ \displaystyle\int\prod_{l=i+1}^{j-1}\mathrm{d}\lambda_l\prod_{l=i}^{j-1}E_{l,l+1}(\lambda_l,\lambda_{l+1}) & \text{if } i<j-1.\end{cases}\tag{1.20}$$

In the $p=1$ case discussed in [12] there were only four kernels (the K_{ij}); indeed, the E_{ij} which were just numbers were absorbed into the normalizations. However, in the general case, the E_{ij} contain integrations and cannot be absorbed. Note that the E_{ij} are the propagators from ψ_i to ψ_j $(i<j)$:

$$\int \mathrm{d}\lambda_j\,\tilde{\psi}_{j;n}(\lambda_j)E_{i,j}(\lambda_i,\lambda_j)=\tilde{\psi}_{i;n}(\lambda_i)$$

$$\int \mathrm{d}\lambda_i\,\psi_{i;n}(\lambda_j)E_{i,j}(\lambda_i,\lambda_j)=\psi_{j;n}(\lambda_j).$$

We thus have the following projection relations:

$$\begin{aligned}&\int \mathrm{d}\lambda_j\,E_{ij}E_{jl}=E_{il} \qquad \text{if } i<j<l\\ &\int \mathrm{d}\lambda_j\,E_{ij}K_{lj}=K_{li} \qquad \text{if } i<j\\ &\int \mathrm{d}\lambda_i\,E_{ij}K_{il}=K_{jl} \qquad \text{if } i<j\\ &\int \mathrm{d}\lambda_j\,K_{ij}K_{jl}=\frac{1}{N}K_{il}.\end{aligned}\tag{1.21}$$

1.6. Correlation functions

In terms of these kernels, the joint density (1.2) of all the eigenvalues of all the matrices can be rewritten:

$$\rho=\text{cte}\,\det K_{0,p}\det E_{0,1}\det E_{1,2}\ldots\det E_{p-1,p}.$$

To obtain the densities and correlation functions of some set of eigenvalues, we have to partially integrate ρ with respect to the other eigenvalues, and this can be done [15] with the help of the projection rules (1.21). The general result is given in appendix A. Here, we will only consider the one- and two-point functions.

The density of eigenvalues (the one-point function) of the ith matrix is

$$\rho_i(\lambda_i)=K_{i,i}(\lambda_i,\lambda_i)\tag{1.22}$$

and the two-point connected correlation function of one eigenvalue λ_i of the ith matrix and one eigenvalue μ_j of the jth matrix is

$$\rho^{(c)}_{i,j}(\lambda_i,\mu_j)=-K_{i,j}(\lambda_i,\mu_j)K_{j,i}(\mu_j,\lambda_i)+\frac{1}{N}K_{i,j}(\lambda_i,\mu_j)E_{i,j}(\lambda_i,\mu_j)\qquad(i\leqslant j).\tag{1.23}$$

We now have to evaluate the kernels K_{ij} and E_{ij} in the large N limit. The first step will be a generalization of the Darboux–Christoffel theorem, which allows one to rewrite K_{ij} as a sum of a small number of terms, instead of the sum of N terms as in (1.19). The second step will be to use the WKB approximations (1.18) for the ψ's. The propagators E_{ij} will be evaluated by steepest descent.

1.7. Generalization of the Darboux–Christoffel theorem for the kernels

As in [12] the Darboux–Christoffel theorem can be generalized. Formally, we write that

$$\psi_{i;n}(\lambda_0) = \hat{x}^{-(N-n)}\psi_{i;N}(\lambda_0) \qquad \tilde{\psi}_{j;n}(\lambda_p) = \hat{x}^{N-n}\tilde{\psi}_{j;N}(\lambda_p) \tag{1.24}$$

and we sum up the geometrical series in (1.19):

$$K_{i,j}(\lambda_i, \lambda_j) = \frac{1}{N}\frac{\hat{y}}{\hat{x}-\hat{y}}\psi_{i;N}(\lambda_i(x))\tilde{\psi}_{j;N}(\lambda_j(y))) \tag{1.25}$$

(we have called $\hat{y}$ the operator $\hat{x}$ acting on the second variable). Multiplying both sides of (1.25) by $(\lambda_i(\hat{x}) - \lambda_i(\hat{y}))$ would give on the left-hand side a differential polynomial acting on K_{ij} (indeed $\lambda_i(\hat{y})$ can be rewritten as a polynomial in $\hat{\lambda}_j$ and $\hat{P}_j$ with the help of equations (1.7) and (1.9)), and on the right-hand side a polynomial in $\hat{x}$ and $\hat{y}$, i.e. a small number of $\psi_{i;n}$ and $\tilde{\psi}_{j;n}$ with $|n - N| \ll N$ (an explicit example is given in appendix C). However, we will not do this, but use (1.25) directly in the large N limit, where $\hat{x}$ and $\hat{y}$ become numbers x and y.

The kernels can thus be approximated by

$$K_{i,j}(\lambda_i, \lambda_j) \sim \frac{1}{N}\frac{y}{x-y}\psi_{i;N}(\lambda_i(x))\tilde{\psi}_{j;N}(\lambda_j(y))$$

and using the WKB asymptotic expressions (1.18) of $\psi_{i;N}$ and $\tilde{\psi}_{j;N}$

$$K_{i,j} \sim \sum_{xy}\left(\frac{\pi}{Nc}\right)^{(i-j)/2}\frac{1}{2\mathrm{i}\pi N}\frac{1}{x-y}\frac{1}{\sqrt{\lambda_i'(x)\lambda_j'(y)}} \times \exp\left[-Nc\left(\int^x(\lambda_{i+1}-\lambda_{i-1})\lambda_i' - \int^y(\lambda_{j+1}-\lambda_{j-1})\lambda_j'\right)\right] \tag{1.26}$$

where x and y appear in complex conjugate pairs, solutions of $\lambda_i(x) = \lambda_i$ and $\lambda_j(y) = \lambda_j$.

One can also find an asymptotic expression for the kernel E_{ij} by steepest descent:

$$E_{ij} \sim \left(\frac{\pi}{Nc}\right)^{(j-i-1)/2}\frac{1}{\sqrt{D_{ij}(x,y)}}\exp[-NcU_{ij}(\lambda_i, \lambda_j)] \tag{1.27}$$

where

$$U_{ij}(\lambda, \mu) = \sum_{l=i}^{j-1}\left((\lambda_l - \lambda_{l+1})^2 + \frac{1}{2c}V_l(\lambda_l) + \frac{1}{2c}V_{l+1}(\lambda_{l+1})\right).$$

$\lambda_l(\lambda, \mu)$ are determined by the saddle-point equation

$$\lambda_i = \lambda \qquad \lambda_j = \mu \qquad 2\lambda_l + \frac{1}{2c}V_l'(\lambda_l) = \lambda_{l-1} + \lambda_{l+1} \qquad \text{for } i < l < j$$

and D_{ij} is the determinant of the matrix of the second derivatives of U_{ij} with respect to the λ_l's:

$$D_{ij} = \det\begin{vmatrix} 2+\frac{V_{i+1}''(\lambda_{i+1})}{2c} & -1 & 0 & \dots & 0 & 0 \\ -1 & 2+\frac{V_{i+2}''(\lambda_{i+2})}{2c} & -1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & 0 & -1 & 2+\frac{V_{j-2}''(\lambda_{j-2})}{2c} & -1 \\ 0 & 0 & \dots & 0 & -1 & 2+\frac{V_{j-1}''(\lambda_{j-1})}{2c} \end{vmatrix}$$

In the particular case $x = y$ we have

$$U_{ij}(x,x) = -\int^x (\lambda_{i+1} - \lambda_{i-1})\lambda_i' + \int^x (\lambda_{j+1} - \lambda_{j-1})\lambda_j'$$

and

$$D_{ij}(x,x) = \sum_{l=i}^{j-1} \frac{\lambda_i'(x)\lambda_j'(x)}{\lambda_l'(x)\lambda_{l+1}'(x)}.$$

Substituting (1.26) and (1.27) into (1.22) and (1.23) we can now evaluate the correlation functions.

1.8. Correlation functions in the short-distance limit

Case $i = j$
Setting $i = j$ in (1.26) gives

$$K_{i,i}(\lambda,\mu) \sim \sum_{xy} \frac{1}{2\mathrm{i}\pi N} \frac{1}{x-y} \frac{1}{\sqrt{\lambda_i'(x)\lambda_i'(y)}} \exp\left[-Nc \int_y^x (\lambda_{i+1} - \lambda_{i-1})\lambda_i' \right]$$

where x and y are the complex solutions of $\lambda_i(x) = \lambda$ and $\lambda_i(y) = \mu$. When λ is close to μ, to leading order, we keep only the values of x and y such that $|x - y|$ is small. This then reduces to

$$K_{i,i}(\lambda,\mu) \sim -\frac{1}{\pi N} \frac{1}{\lambda - \mu} \sin\{Nc(\lambda-\mu)\mathrm{Im}\,(\lambda_{i+1} - \lambda_{i-1})\} \times \exp[-Nc(\lambda-\mu)\mathrm{Re}\,(\lambda_{i+1} - \lambda_{i-1})].$$

In particular, when $\lambda = \mu$ we obtain the density

$$\rho_i(\lambda) = K_{i,i}(\lambda,\lambda) = -\frac{c}{\pi}\mathrm{Im}\,(\lambda_{i+1}(x) - \lambda_{i-1}(x)) = -\frac{1}{\pi}\mathrm{Im}\,P_i(x). \tag{1.28}$$

When λ is close to μ but different, we can compute the two-point connected correlation function

$$\rho_{i,i}^{(c)}(\lambda,\mu) = -K_{i,i}(\lambda,\mu)K_{i,i}(\mu,\lambda)$$

i.e.

$$\rho_{i,i}^{(c)}(\lambda,\mu) \underset{\lambda\to\mu}{\sim} -\left(\frac{\sin N\pi(\lambda-\mu)\rho_i(\lambda)}{N\pi(\lambda-\mu)}\right)^2 \tag{1.29}$$

and we recover the universal two-point correlation function in the short-distance regime.

Case $i \neq j$
It is now meaningless to consider the limit λ_i close to λ_j since they are eigenvalues of different matrices. Generically, K_{ij} is of order $1/N$, which means that the connected correlation function is of order $1/N^2$, and we can say that in the large N limit λ_i and λ_j are uncorrelated.

The only limit in which the correlation may become larger than $1/N^2$ is the case where $x - y$ becomes small. The equation $x = y$ defines a function $\lambda_i(\lambda_j)$. The problem is that this function takes complex values in the interesting domain—for example, we see from equation (1.28) that $\lambda_{i+1}(\lambda_i)$ or $\lambda_{i-1}(\lambda_i)$ take complex values (this fact has already been debated in [12])—and we have not found any physical interpretation, except in the case of the continuous model described in the next section.

However, let us assume that $|x-y|$ is small (we will also assume $i<j$). We introduce the scaling variable

$$\Delta=\sqrt{N}(x-y)\sqrt{\lambda_i'\lambda_j'}\sqrt{\frac{c}{D_{ij}}}.$$

In the limit $|x-y|\to 0$, the Taylor expansion of the term appearing in the exponential in (1.26) gives

$$\int^x(\lambda_{i+1}-\lambda_{i-1})\lambda_i'-\int^y(\lambda_{j+1}-\lambda_{j-1})\lambda_j'=-U_{ij}+\frac{\lambda_i'\lambda_j'}{D_{ij}}(x-y)^2+\mathrm{O}((x-y)^3).$$

Therefore, we have

$$\begin{aligned}
K_{ij}&\sim\left(\frac{\pi}{Nc}\right)^{(i-j)/2}\frac{1}{2\mathrm{i}\pi\sqrt{N}}\sqrt{\frac{c}{D_{ij}}}\,\mathrm{e}^{+NcU_{ij}}\frac{1}{\Delta}\,\mathrm{e}^{-\Delta^2}\\
K_{ji}&\sim\left(\frac{\pi}{Nc}\right)^{(j-i)/2}\frac{1}{2\mathrm{i}\pi\sqrt{N}}\sqrt{\frac{c}{D_{ij}}}\,\mathrm{e}^{-NcU_{ij}}\frac{1}{\Delta}\,\mathrm{e}^{\Delta^2}\\
\frac{1}{N}E_{ij}&\sim\left(\frac{\pi}{Nc}\right)^{(j-i)/2}\frac{1}{\sqrt{\pi}\sqrt{N}}\sqrt{\frac{c}{D_{ij}}}\,\mathrm{e}^{-NcU_{ij}}.
\end{aligned}\tag{1.30}$$

Remember that each of these expressions is actually a sum over the different values of x and y satisfying $\lambda_i(x)=\lambda$ and $\lambda_j(y)=\mu$. However, since λ and μ are not assumed real, the different values of x or y contributing to this sum are not the complex conjugates of each other, and the exponential term cannot simply be replaced by a sine function.

However, we have the correlation function

$$\rho_{i,j}^{(c)}\underset{\Delta\to 0}{\sim}\sum_{xy}-\frac{1}{4\pi^2N}\frac{c}{D_{ij}}\frac{1}{\Delta}\,\mathrm{e}^{NcU_{ij}}\sum_{xy}\left(\frac{1}{\Delta}-2\sqrt{\pi}\,\mathrm{e}^{-\Delta^2}\right)\mathrm{e}^{-NcU_{ij}}.\tag{1.31}$$

To go further one would have to make some assumption on the argument of x and y (i.e. where in the complex plane are λ and μ), and then separate the imaginary and real parts of U_{ij} and Δ. We would then observe that in this scaling regime ($|x-y|\sim 1/\sqrt{N}$), $\rho_{i,j}^{(c)}$ is of order $\mathrm{O}(1/N)$ instead of $\mathrm{O}(1/N^2)$, and that it presents some kind of universality: it does not depend very explicitly on the potentials $V_i(\lambda)$. The signification of this correlation function is therefore not clear, and has to be further studied. However, this calculation was performed to prepare for the next section, the continuous chain of matrices, where it is possible to have both λ and μ real and $(x-y)$ small, because D_{ij} will be small too.

1.9. Smoothed functions

It has been argued that once the short-distance oscillations of the correlation functions have been smoothed out, they should present some universality properties, i.e. they do not depend on the potentials V_i (cf [8] for the one-matrix model).

Indeed, when we smooth expression (1.23) (we keep only the terms which do not oscillate at frequency N in the sums over x and y, i.e. those for which the exponentials exactly cancel), we obtain

$$\rho_{i,j}^{(c)}(\lambda_i,\lambda_j)_{\mathrm{smooth}}\sim-\frac{1}{4\pi^2N^2}\frac{x'(\lambda_i)y'(\lambda_j)}{(x(\lambda_i)-y(\lambda_j))^2}+\text{ complex conjugate.}\tag{1.32}$$

This result is exactly the same as for two matrices [7, 12, 17] and recalls the universal result of [8].

2. Continuum limit

The model of a chain of matrices is naturally extended to the model of a time-dependent random matrix. We replace the integer index i by a continuous time $t = i\epsilon$, which runs from 0 to $T = p\epsilon$. The coupling term $c(M_i - M_{i+1})^2$ becomes a kinetic term and we set

$$c = \frac{1}{2\epsilon} \qquad \sum_i \longrightarrow \frac{1}{\epsilon}\int \mathrm{d}t \qquad V_i(\lambda) \longrightarrow \epsilon V(\lambda, t).$$

The partition function thus becomes a functional integral for a one-dimensional matrix field:

$$Z = \int D[M(t)] \exp\left[-N\mathrm{tr}\int_0^T \mathrm{d}t\left(\frac{1}{2}\left(\frac{\mathrm{d}M}{\mathrm{d}t}\right)^2 + V(M(t), t)\right)\right].$$

Time-dependent or parameter-dependent random matrices represent out of equilibrium or more realistic physical systems, and have been much studied [18, 19]. The usual observables are the correlations of velocities, and the curvatures (second derivatives with respect to time), which can probably be derived in the context of the results presented here but which we shall consider now, are just the limits of the results of section 1, i.e. the two-point correlation functions.

We now have a set of orthogonal wavefunctions depending on time

$$\psi_{i;n}(\lambda) \longrightarrow \psi_n(\lambda, t) \qquad \text{and} \qquad \tilde{\psi}_{i;n}(\lambda) \longrightarrow \tilde{\psi}_n(\lambda, t)$$

which satisfy the orthogonality relationship

$$\int \mathrm{d}\lambda\, \psi_n(\lambda, t)\tilde{\psi}_m(\lambda, t) = \delta_{n,m}.$$

In fact, one has to change the normalization, just to absorb the constant factor $(\pi/Nc)^{i/2}$ of equation (1.18) which becomes infinite in the continuum limit. However, this does not change anything else.

2.1. The function λ(x,t)

We can define a family of functions $\lambda(x, t)$ of an auxiliary variable x:

$$\lambda_i(x) \longrightarrow \lambda(x, t) = \sum_k \alpha_k(t)x^k. \tag{2.1}$$

Note that according to equation (1.13), k runs from $-\infty$ to $+\infty$ (except for the Gaussian case), so that (2.1) has to be taken as a formal expansion in the Laurent series.

$\lambda(x, t)$ obeys the continuous limit of the equations of motion (1.15)

$$\ddot{\lambda} = \frac{\partial^2}{\partial t^2}\lambda(x, t) = V'(\lambda(x, t)) \tag{2.2}$$

with the boundary conditions (1.16)

$$\begin{aligned} &\lambda(x, 0) = \alpha x + \sum_{k\leqslant 0}\alpha_k(0)x^{-k} \qquad \dot{\lambda}(x, 0) = -\frac{1}{x\alpha} + \mathrm{O}(1/x^2) \\ &\lambda(x, T) = \frac{1}{x}\alpha + \sum_{k\geqslant 0} x^k\alpha_k(T) \qquad \dot{\lambda}(x, T) = \frac{x}{\alpha} + \mathrm{O}(x^2). \end{aligned} \tag{2.3}$$

These last conditions can be rewritten in a way which does not involve any expansion in powers of x†:

$$\dot\lambda(x,0) \sim -\frac{1}{\lambda(x,0)} \qquad \text{when } \lambda(x,0) \to \infty$$
$$\dot\lambda(x,T) \sim \frac{1}{\lambda(x,T)} \qquad \text{when } \lambda(x,T) \to \infty.$$

However, to define the function $x(\lambda,t)$ you cannot avoid performing the formal expansion.

Remark. Equation (1.14) becomes $\sum_k k\dot\alpha_k\alpha_{-k} = 1$.

2.2. The momenta

The momentum $P(x,t)$ is the time derivative of $\lambda(x,t)$ (equations (1.7) and (1.8)):

$$P(t) = \dot\lambda = \left.\frac{\partial\lambda}{\partial t}\right|_x = v(\lambda,t) - \mathrm{i}\pi\rho(\lambda,t).$$

Its imaginary part $\rho(\lambda,t) = (-1/\pi)\mathrm{Im}\,P(t)$ is the density of eigenvalues at time t.

2.3. The kernels

The discrete kernels K_{ij} and E_{ij} naturally have some continuous version:

$$K_{i,j}(\lambda,\mu) \longrightarrow K(\lambda,t|\mu,t') = \frac{1}{N}\sum_{n=0}^{N-1}\psi_n(\lambda,t)\tilde\psi_n(\mu,t')$$
$$E_{i,j}(\lambda,\mu) \longrightarrow E(\lambda,t|\mu,t') = \int_{\lambda(t)=\lambda}^{\lambda(t')=\mu} D[\lambda(\tau)]\exp\left[-N\int_t^{t'}\mathrm{d}\tau\left(\frac{1}{2}\dot\lambda^2 + V(\lambda(\tau),\tau)\right)\right].$$

E is the usual quantum mechanics propagator for a single particle in the potential V:

$$\psi_n(\lambda,t') = \int \mathrm{d}\mu\,\psi_n(\mu,t)E(\mu,t|\lambda,t')$$

and these kernels allow us to calculate the two-point connected correlation function ($t < t'$):

$$\rho^{(c)}(\lambda,t|\mu,t') = -K(\lambda,t|\mu,t')\left(K(\mu,t'|\lambda,t) - \frac{1}{N}E(\lambda,t|\mu,t')\right).$$

2.4. Correlations

Let us calculate the two-point correlation in the limit where $\delta\lambda = \lambda - \mu$ and $\delta t = t - t'$ are small, of order $1/N$. In this purpose, we have to evaluate the kernel $K(\lambda,t|\lambda+\delta\lambda,t+\delta t)$ with the WKB approximation (1.26) (remember that we have absorbed the (π/Nc) factors):

$$K \sim 2\mathrm{Re}\,\frac{1}{2\mathrm{i}\pi N}\frac{1}{\sqrt{\lambda'\lambda'}}\frac{-1}{\delta x}\exp\left[-N\left(\int^{\lambda}P(\lambda',t)\,\mathrm{d}\lambda' - \int^{\lambda+\delta\lambda}P(\lambda',t+\delta t)\,\mathrm{d}\lambda'\right)\right]$$

† Even in the discrete case, $\omega_0(\lambda_0(x)) = 2c(\lambda_0(x) - \lambda_1(x)) + V_0'(\lambda_0(x))$ is the resolvent of the first matrix $\omega_0(\lambda) = (1/N)\langle\mathrm{Tr}(1/(\lambda - M_0))\rangle$, and $\omega_p(\lambda_p(x)) = 2c(\lambda_p(x) - \lambda_{p-1}(x)) + V_p'(\lambda_p(x))$ is the resolvent of the pth matrix. The resolvent behaves as $\omega(\lambda) \sim 1/\lambda$ at large λ. In the continuum limit, we have $\omega(\lambda(x,0),0) = -\dot\lambda(x,0)$ and $\omega(\lambda(x,T),T) = \dot\lambda(x,T)$. For the intermediate values of i (intermediate times), there is no simple relationship between the resolvent $\omega_i(\lambda)$ ($\omega(\lambda,t)$) and the functions $\lambda_i(x)$ ($\lambda(x,t)$).

where $P(\lambda,t)=\dot{\lambda}=(\partial\lambda/\partial t)|_x = v-\mathrm{i}\pi\rho$. Writing $\delta\lambda=\lambda'\delta x+P\delta t$, the denominator $\lambda'\delta x$ is $\delta\lambda - v\delta t + \mathrm{i}\pi\rho\delta t$, and the term in the exponential is $P\delta\lambda+\delta t\int^{\lambda}(\partial P/\partial t)|_\lambda$ which is evaluated through a play with the partial derivatives at λ fixed or x fixed:

$$V'(\lambda)=\ddot{\lambda}=\left.\frac{\partial P}{\partial t}\right|_x=\left.\frac{\partial P}{\partial t}\right|_\lambda+\left.\frac{\partial P}{\partial\lambda}\right|_t\left.\frac{\partial\lambda}{\partial t}\right|_x.$$

We thus obtain

$$K\sim \mathrm{Im}\,\frac{1}{N\pi(v\delta t-\delta\lambda-\mathrm{i}\pi\rho\delta t)}\exp[N\mathrm{i}\pi\rho(v\delta t-\delta\lambda)]\times\exp\left[N\left(v\delta\lambda+\delta t\left(V-\frac{1}{2}v^2+\frac{1}{2}\pi^2\rho^2\right)\right)\right]. \tag{2.4}$$

Similarly, observing that the continuous limit of D_{ij} is

$$\frac{1}{c}D_{ij}\sim 2\delta t$$

the asymptotic expression (1.27) for E turns into

$$\frac{1}{N}E\sim\frac{1}{\sqrt{2\pi N\delta t}}\exp\left[-N\left(\frac{1}{2}\frac{\delta\lambda^2}{\delta t}+V\delta t\right)\right]\qquad\text{for }\delta t>0. \tag{2.5}$$

• Equal time correlations. When $\delta t=0$ we recover the usual one-matrix model result:

$$K\sim\frac{\sin(N\pi\rho\delta\lambda)}{N\pi\delta\lambda}\,\mathrm{e}^{Nv\delta\lambda}.$$

For instance, when $\delta\lambda=0$ we obtain the one-point function (the density) $K=\rho$. When $\delta\lambda\neq 0$ but of order $1/N$, we have the universal correlation function:

$$\rho^{(c)}(\lambda,t|\lambda+\delta\lambda,t)\sim-\rho^2\left(\frac{\sin(N\pi\rho\,\delta\lambda)}{N\pi\rho\delta\lambda}\right)^2.$$

Therefore, at equal times, the correlation function of close eigenvalues is still universal.

• Different times. When $\delta t\neq 0$, the denominator of K never vanishes. It reaches its minimum for $\delta\lambda=v\delta t$. We shall thus consider the regime

$$\delta\lambda=v\delta t\qquad\text{and}\qquad\delta t\sim 1/N$$

where we have

$$K\sim\frac{\rho}{N\pi^2\rho^2\delta t}\exp\left[N\delta t\left(V+\frac{1}{2}v^2+\frac{1}{2}\pi^2\rho^2\right)\right]$$

and

$$\frac{1}{N}E\sim\frac{\rho\sqrt{\pi}}{\sqrt{2N\pi^2\rho^2\delta t}}\exp\left[-N\delta t\left(V+\frac{1}{2}v^2\right)\right].$$

Thus

$$\rho^{(c)}(\lambda,t|\lambda+v\delta t,t+\delta t)\sim-\rho^2\frac{1}{\Delta}\left(\frac{1}{\Delta}+\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{\frac{1}{2}\Delta}}{\sqrt{\Delta}}\right)\qquad\text{with }\Delta=N\pi^2\rho^2\delta t \tag{2.6}$$

which is remarkably universal; it does not depend on the precise form of the potential $V(\lambda,t)$.

3. Conclusions

We have thus, as announced in [12], generalized the properties of the correlation functions of the two-matrix model to the chain of matrices.

As expected, the chain of matrices obeys the same universal properties as the one- and two-matrix models. We obtain similar results.

• The correlation function is generically of order $1/N^2$, except in the regime where the denominator of the kernel K becomes small. This may happen only at a small distance $\delta\lambda$ and at a small time interval δt.

• At equal time ($\delta t = 0$) we recover exactly the well known universal correlation function of the one-matrix model. This is therefore one more matrix model where this universality is proved.

• At different times ($\delta t > 0$), the two-point connected correlation function is again a universal function of δt and the covariant interval $\delta\lambda - v\delta t$, i.e. it does not depend on the potential $V(\lambda, t)$ (indeed the term involving V exactly cancels when multiplying (2.4) and (2.5)).

Moreover, we have given a very explicit way to compute the correlation functions. This method (the generalized Darboux–Christoffel theorem) might allow one to compute the correlation functions even for N finite, provided one knows the orthogonal polynomials, although what we have computed here is an approximation in the large N limit, using the WKB approximations of the orthogonal polynomials.

This method has shown that the function $x(\lambda)$ plays a very important role; as the best auxiliary variable, its physical meaning should be better understood, beyond the orthogonal polynomial's method. For instance, another way of solving the chain of matrices is through the loop equations (the invariance of the partition function with respect to the change of variables, see appendix D). The loop equations exist even for models which have no orthogonal polynomial methods, and enable one to compute the subleading terms in the $1/N$ expansion. These equations (in the large N limit) can be written (appendix D) as an algebraic equation (of very high degree) for the resolvent of the first (or last) matrix of the chain. The parametrization of λ and $\omega(\lambda)$ as functions of x allows us to solve this algebraic equation, or at least to reduce drastically its degree. This is an indication that the function $x(\lambda)$ is a very good variable to deal with the loop equations, and maybe that $x(\lambda)$ could be defined out of the orthogonal polynomial's frame. All this is reminiscent of the diagrammatic method of [8] to derive the two-point correlation function.

We have not considered here the smoothed k-point connected correlation functions with $k > 2$, in the long-distance regime, for which [7, 8, 12] claim some universality properties. The reason is that the orthogonal polynomials method fails in this case. Indeed, the k-point connected correlation function is a product of k kernels, and thus is of order N^{-k}, while it is well known from quantum gravity that the smoothed part is of order N^{2-2k} (it can be understood topologically [7, 12]: the exponent of N is the Euler characteristic of k disks for the leading disconnected term, and of a sphere with k holes for the connected part). In the one-matrix case [7], the smoothed k-point correlation functions (with $k > 2$) are derived by the loop equations or equivalently by diagrammatic methods [8].

We have considered here only the Hermitian matrix case, although orthogonal polynomial methods are also available in the orthogonal and symplectic cases, with some refinements such as the introduction of pfaffians. It is thus likely that some of the results presented here can have consequences for the non-Hermitian models (the WKB approximations and the generalized Darboux–Christoffel theorem). The problem is that the Itzykson–Zuber formula cannot be generalized, and the whole model of the chain of matrices requires a completely different approach.

Another generalization one might think about would be a closed chain with, for instance, periodic boundary conditions ($M_p = M_0$), but for the same reason (the Itzykson–Zuber formula), the orthogonal polynomial method does not hold.

The main prospect should now be to study the correlations of velocities and curvatures.

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Appendix A. Correction functions of any set of eigenvalues

The general correlation function of any set of eigenvalues has been computed in [15] (see also [13] for the one-matrix case).

In addition to the kernels K_{ij} and E_{ij}, we will need to introduce the H_{ij}'s:

$$H_{ij} = K_{ij} - \frac{1}{N}E_{ji} \qquad \text{if } j \leqslant i \qquad H_{ij} = K_{ij} \qquad \text{if } j > i.$$

To compute a partial density of eigenvalues, for instance

$$\rho_{n_0,\dots,n_p}(\lambda_{0;1},\dots,\lambda_{0;n_0},\dots,\lambda_{p;1},\dots,\lambda_{p;n_p})$$

you have to consider all the possible permutations of all the $n = n_0 + \cdots + n_p$ variables, and then (with obvious notations)

$$\rho(\lambda_1,\dots,\lambda_n) = \det_{i,j} H_{i,j} = \sum_{\sigma}(-1)^{\sigma}\prod_i H_{i,\sigma(i)}(\lambda_i,\lambda_{\sigma(i)}).$$

The connected correlations are obtained by reducing the sum to the cyclic permutations only.

For instance, we have (we assume $i \leqslant j \leqslant k$)

$$\rho_i(\lambda_i) = K_{i,i}(\lambda_i,\lambda_i)$$

$$\rho^{(c)}_{i,j}(\lambda_i,\lambda_j) = -K_{i,j}(\lambda_i,\lambda_j)K_{j,i}(\lambda_j,\lambda_i) + \frac{1}{N}K_{i,j}(\lambda_i,\lambda_j)E_{i,j}(\lambda_i,\lambda_j)$$

$$\rho^{(c)}_{i,j,k} = K_{ij}K_{jk}K_{ki} + K_{ik}K_{kj}K_{ji} - \frac{1}{N}(K_{ij}K_{jk}E_{ik} + K_{ji}K_{ik}E_{jk} + K_{ik}K_{kj}E_{ij}) + \frac{1}{N^2}K_{ik}E_{ij}E_{jk}.$$

Appendix B. Explicit expressions of orthogonal polynomials

The orthogonal polynomials $\mathcal{P}_n$ and $\tilde{\mathcal{P}}_n$ have explicit expressions as matrix integrals:

$$\mathcal{P}_n(\lambda) = \int \mathrm{d}M_{0\,n\times n}\dots\mathrm{d}M_{p\,n\times n}\det(\lambda - M_0)\prod_{i=0}^{p}\exp[-N\,\mathrm{tr}\,V_i(M_i)] \times\prod_{i=0}^{p-1}\exp[-Nc\,\mathrm{tr}(M_i - M_{i+1})^2]$$

$$\tilde{\mathcal{P}}_n(\tilde{\lambda}) = \int \mathrm{d}M_{0\,n\times n}\dots\mathrm{d}M_{p\,n\times n}\det(\tilde{\lambda} - M_p)\prod_{i=0}^{p}\exp[-N\,\mathrm{tr}\,V_i(M_i)] \times\prod_{i=0}^{p-1}\exp[-Nc\,\mathrm{tr}(M_i - M_{i+1})^2].$$

Appendix C. An explicit example: the Gaussian potential, $V(\lambda) = \frac{1}{2}g\lambda^2$

C.1. Discrete case

Let us consider a constant Gaussian potential: $V_i'(\lambda) = g\lambda$. We will use the parametrization $g = 4c(\cosh\chi - 1)$ for later convenience.

The operators $\hat{\lambda}_i$ contain only two terms, which we write as

$$\hat{\lambda}_i = \alpha_i \hat{x}\sqrt{\frac{\hat{n}}{N}} + \beta_i \sqrt{\frac{\hat{n}}{N}}\frac{1}{\hat{x}}.$$

The equation of motions are linear and thus easily solved:

$$\alpha_{i+1} + \alpha_{i-1} - 2\alpha_i = \frac{g}{2c}\alpha_i \qquad \beta_{i+1} + \beta_{i-1} - 2\beta_i = \frac{g}{2c}\beta_i$$

with the boundary conditions:

$$\alpha_0 = \beta_p = \alpha \qquad \alpha_1 = \left(1 + \frac{g}{2c}\right)\alpha \qquad \beta_{p-1} = \left(1 + \frac{g}{2c}\right)\alpha$$
$$\left(1 + \frac{g}{2c}\right)\alpha_p - \alpha_{p-1} = \frac{1}{2c\alpha}.$$

The first line implies that $\beta_i = \alpha_{p-i}$. The solution is then

$$\alpha_i = \alpha \frac{\cosh\left(i + \frac{1}{2}\right)\chi}{\cosh\frac{1}{2}\chi} \qquad \text{where} \qquad \alpha^2 = \frac{1}{4c\sinh(p+1)\chi\tanh\frac{1}{2}\chi}.$$

The momentum $\hat{P}_i = (1/N)\mathrm{d}/\mathrm{d}\lambda_i$ is

$$\hat{P}_i^\dagger = -2\frac{1 - A_i}{a_i^2}\hat{\lambda}_i^\dagger + \frac{1}{\alpha_i}\sqrt{\frac{\hat{n}}{N}}\frac{1}{\hat{x}} \qquad \hat{P}_i = -2\frac{1 + A_i}{a_i^2}\hat{\lambda}_i + \frac{1}{\alpha_{p-i}}\hat{x}\sqrt{\frac{\hat{n}}{N}}$$

where we have introduced $A_i = \sinh(p - 2i)\chi / \sinh(p+1)\chi$ and

$$a_i = 2\sqrt{\alpha_i \alpha_{p-i}}$$

which will play a very important role as the natural scale for λ_i. In fact, $2a_i$ is the width of the distribution of eigenvalues of the ith matrix. Indeed, in the large N limit and $n \sim N$, we can eliminate x and write P_i as a function of λ_i:

$$P_i(\lambda_i) = -\frac{2}{a_i}\left(A_i\frac{\lambda_i}{a_i} - \sqrt{\frac{\lambda_i^2}{a_i^2} - 1}\right).$$

Its imaginary part is the density of eigenvalues of the ith matrix,

$$\rho_i(\lambda_i) = \frac{2}{\pi a_i}\sqrt{1 - \frac{\lambda_i^2}{a_i^2}}$$

which is a semi-circle law of diameter $2a_i$.

The wavefunctions can be exactly computed in terms of Hermite polynomials $H_n(x)$:

$$\psi_{i;n}(\lambda_i) = c_i\left(\frac{\alpha_{p-i}}{\alpha_i}\right)^{n/2}\frac{1}{\sqrt{n!}}H_n\left(2\sqrt{N}\frac{\lambda_i}{a_i}\right)\exp\left[-N(1 - A_i)\frac{\lambda_i^2}{a_i^2}\right]$$
$$\tilde{\psi}_{j;n}(\lambda_j) = \tilde{c}_j\left(\frac{\alpha_j}{\alpha_{p-j}}\right)^{n/2}\frac{1}{\sqrt{n!}}H_n\left(2\sqrt{N}\frac{\lambda_j}{a_j}\right)\exp\left[-N(1 + A_j)\frac{\lambda_j^2}{a_j^2}\right]$$

(where c_i and $\tilde{c}_j$ are some constants, irrelevant for what we need; we just know that $c_i\tilde{c}_i = (1/a_i)\sqrt{2N/\pi}$).

We can write some WKB approximations in the large N limit:

$$\psi_{i;n}(\lambda_i) \sim \left(\frac{\pi}{Nc}\right)^{(i/2)-(p/4)} \frac{2(2\pi)^{-1/4}}{\sqrt{a_i \sin\phi_i}} \left(\frac{\alpha_{p-i}}{\alpha_i}\right)^{(2n-2N+1)/4} \times \cos\left[\left(n+\frac{1}{2}\right)\phi_i - \frac{N}{2}\sin 2\phi_i - \frac{\pi}{4}\right] \exp\left[NA_i \frac{\lambda_i^2}{a_i^2}\right]$$

$$\tilde{\psi}_{j;n}(\lambda_i) \sim \left(\frac{\pi}{Nc}\right)^{(p/4)-(j/2)} \frac{2(2\pi)^{-1/4}}{\sqrt{a_j \sin\phi_j}} \left(\frac{\alpha_j}{\alpha_{p-j}}\right)^{(2n-2N+1)/4} \times \cos\left[\left(n+\frac{1}{2}\right)\phi_j - \frac{N}{2}\sin 2\phi_j - \frac{\pi}{4}\right] \exp\left[-NA_j \frac{\lambda_j^2}{a_j^2}\right]$$

where $\lambda_i = a_i \cos\phi_i$.

Now let us compute the kernels. The propagator E_{ij} is

$$E_{i,j}(\lambda_i, \lambda_j) = \left(\frac{\pi}{Nc}\right)^{(j-i-1)/2} \sqrt{\frac{\sinh\chi}{\sinh(j-i)\chi}} \times \exp\left\{-Nc\frac{\sinh\chi}{\sinh(j-i)\chi}[\cosh(j-i)\chi(\lambda_i^2+\lambda_j^2) - 2\lambda_i\lambda_j]\right\}.$$

This is an exact result since the saddle-point method is exact in the Gaussian case. We also have the determinant of the second derivatives of the potential $D_{ij}(\lambda_i, \lambda_j)$ which is a constant:

$$D_{ij} = \frac{\sinh(j-i)\chi}{\sinh\chi}.$$

The kernel $K_{ij}(\lambda_i, \lambda_j)$ obeys a generalization of the Darboux–Christoffel theorem:

$$\left\{2\frac{\lambda_i}{a_i} - 2\tau\frac{\lambda_j}{a_j} + \left(\tau - \frac{1}{\tau}\right)\left((1+A_j)\frac{\lambda_j}{a_j} + \frac{a_j}{2N}\frac{\partial}{\partial\lambda_j}\right)\right\} K_{ij} = \sqrt{\frac{\alpha_i}{\alpha_{p-i}}}\psi_{i;N}\tilde{\psi}_{j;N-1} - \sqrt{\frac{\alpha_{p-i}}{\alpha_i}}\psi_{i;N-1}\tilde{\psi}_{j;N}$$

$$\left\{2\tau\frac{\lambda_i}{a_i} - 2\frac{\lambda_j}{a_j} - \left(\tau - \frac{1}{\tau}\right)\left((1-A_i)\frac{\lambda_i}{a_i} + \frac{a_i}{2N}\frac{\partial}{\partial\lambda_i}\right)\right\} K_{ij} = \sqrt{\frac{\alpha_j}{\alpha_{p-j}}}\psi_{i;N}\tilde{\psi}_{j;N-1} - \sqrt{\frac{\alpha_{p-j}}{\alpha_j}}\psi_{i;N-1}\tilde{\psi}_{j;N}$$

where τ denotes the ratio $\tau = \sqrt{\alpha_{p-i}\alpha_j/\alpha_i\alpha_{p-j}}$. Let us emphasize that these equations are exact, even for N finite (note that when $i = j$ we have $\tau = 1$ and we recover the usual Darboux–Christofel theorem). With the operatorial notation they are obvious, being just a rewriting of

$$\{\lambda_i(\hat{x}) - \lambda_i(\hat{y})\}K_{ij} = \{\lambda_i(\hat{x}) - \lambda_i(\hat{y})\}\frac{\hat{y}}{\hat{x}-\hat{y}}\psi_{i;N}\tilde{\psi}_{j;N}$$

$$\{\lambda_j(\hat{x}) - \lambda_j(\hat{y})\}K_{ij} = \{\lambda_j(\hat{x}) - \lambda_j(\hat{y})\}\frac{\hat{y}}{\hat{x}-\hat{y}}\psi_{i;N}\tilde{\psi}_{j;N}$$

(if you want to rederive them, be careful since here the partial derivative with respect to λ_i carries on the bra and is a $P_i^\dagger$).

In the large N limit, we have

$$K_{ij} \sim \left(\frac{\pi}{Nc}\right)^{(i-j)/2} \frac{1}{\sqrt{a_i \sin\phi_i a_j \sin\phi_j}} \frac{\sqrt{\tau}}{4\pi N} \exp[N(A_i \cos^2\phi_i - A_j \cos^2\phi_j)]$$
$$\times \left\{ \left(1+\frac{1}{\tau}\right) \sin\frac{1}{2}(\phi_i-\phi_j) \sin N(\zeta_i-\zeta_j) + \left(1-\frac{1}{\tau}\right) \right.$$
$$\left. \times \cos\frac{1}{2}(\phi_i-\phi_j) \cos N(\zeta_i-\zeta_j) \right\} \left\{ \frac{1+\tau^2}{2\tau} - \cos(\phi_i-\phi_j) \right\}^{-1}$$
$$+(\text{same thing with } \phi_j \to -\phi_j)$$

where we have noted $\zeta(\phi) = \phi - \frac{1}{2}\sin 2\phi$.

For $i = j$ we have $\tau = 1$ and we recover

$$K_{ii}(\lambda,\mu) \sim \frac{1}{a_i\sqrt{\sin\phi \sin\tilde{\phi}}} \frac{1}{4\pi N} \left(\frac{\sin N(\zeta(\lambda)-\zeta(\mu))}{\sin((\phi-\tilde{\phi})/2)} \right) \exp\left[\frac{N}{2} \frac{A_i}{a_i^2} (\lambda^2-\mu^2) \right]$$

which gives for $\lambda = \mu$ the density of eigenvalues $\rho_i(\lambda) = (1/\pi a_i)\sin\phi$.

When $i \neq j$, we note that the denominator never vanishes. It is maximum when $\phi_i = \phi_j$, i.e. $\lambda_i/a_i = \lambda_j/a_j$; this is the regime in which one could have a not too small correlation.

C.2. Continuous limit

The continuous limit can be obtained by setting $c = 1/2\epsilon$, $t = i\epsilon$, $T = p\epsilon$, $g = \epsilon\nu^2$, and taking the limit $\epsilon \to 0$.

We find

$$\alpha^2 = \frac{1}{\nu \sinh \nu T}$$
$$\lambda(x,t) = \alpha \left(x \cosh \nu t + \frac{1}{x} \cosh \nu(T-t) \right)$$

which we write as

$$\lambda(\phi,t) = a(t)\cos\phi$$

where $a(t)$ is given by

$$a(t) = 2\alpha\sqrt{\cosh \nu t \cosh \nu(T-t)}.$$

In the small-distance regime ($\delta\lambda$ small and δt small), we have (up to some normalization factors which will cancel when we compute the correlation function)

$$K(\lambda,t|\lambda+\delta\lambda,t+\delta t) \sim \frac{1}{2\pi N} \frac{1}{a\sin\phi} \frac{\delta\phi \sin(N\pi\rho a \sin\phi\delta\phi) + (2/a^2)\delta t \cos(N\pi\rho a \sin\phi\delta\phi)}{2(\delta t^2/a^4) + \frac{1}{2}\delta\phi^2}$$

$$\frac{1}{N}E(\lambda,t|\lambda+\delta\lambda,t+\delta t) \sim \frac{1}{\sqrt{2\pi N\delta t}} \exp\left[-\frac{N}{2} \left(a^2 \sin^2\phi \frac{\delta\phi^2}{\delta t} + \frac{4}{a^2}\cos^2\phi\delta t \right) \right].$$

Thus we have the two-point connected correlation function

$$\rho^{(c)}(\lambda,t|\lambda+\delta\lambda,t+\delta t) = -K(\lambda,t|\lambda+\delta\lambda,t+\delta t)$$
$$\times \left(K(\lambda+\delta\lambda,t+\delta t|\lambda,t) - \frac{1}{N}E(\lambda,t|\lambda+\delta\lambda,t+\delta t) \right)$$

which gives

$$\rho^{(c)} = -\frac{1}{4\pi^2 N^2 a^2 \sin^2\phi}\frac{\delta\phi^2 \sin^2(N\pi\rho a \sin\phi\delta\phi) - (4/a^4)\delta t^2 \cos^2(N\pi\rho a \sin\phi\delta\phi)}{(2(\delta t^2/a^4) + \frac{1}{2}\delta\phi^2)^2}$$
$$+\frac{1}{(2\pi N\delta t)^{3/2}}\frac{\delta t}{a\sin\phi}\frac{\delta\phi \sin(N\pi\rho a \sin\phi\delta\phi) + (2/a^2)\delta t \cos(N\pi\rho a \sin\phi\delta\phi)}{2(\delta t^2/a^4) + \frac{1}{2}\delta\phi^2}$$
$$\times \exp\left[-\frac{N}{2}\left(a^2\sin^2\phi\frac{\delta\phi^2}{\delta t} + \frac{4}{a^2}\cos^2\phi\delta t\right)\right]. \qquad \text{(C.1)}$$

Appendix D. The loop equations

The loop equations are nothing but the consequences of reparametrization invariance in the partition function. For instance, for the one-matrix model [20, 21]

$$Z = \int \mathrm{d}M \exp[-N \operatorname{tr} V(M)].$$

The loop equation is a quadratic equation for the resolvent $\omega(z) = (1/N)\mathrm{tr}(1/(z - M))$,

$$\omega(z)(V'(z) - \omega(z)) = P(z)$$

where $P(z)$ is a polynomial in z, whose coefficients (except the leading one) are undetermined by the loop equations:

$$P(z) = \frac{1}{N}\left\langle \mathrm{tr}\frac{V'(z) - V'(M)}{z - M}\right\rangle = \mathrm{Pol}\,\omega(z)V'(z).$$

The coefficients of P have to be determined by some physical input, such as some assumption about the analyticity structure of ω. The most usual input is the one-cut assumption: the requirement that ω has only one cut in the complex plane, i.e. that the support of the density of eigenvalues is connected, which is expected to be the case for a single-well potential.

The case of the chain of matrices is very similar: the resolvent fulfils an algebraic equation, with an undetermined polynomial on the right-hand side. Here, we will just give the final result, and the recipe to write down the algebraic equation for $\omega(z)$ but without any proof.

Consider the chain of matrices

$$Z = \int \mathrm{d}M_0 \ldots \mathrm{d}M_p \exp\left[- N \operatorname{tr}\sum_i U_i(M_i) - 2c\sum_i M_i M_{i+1}\right]$$

(note that the potentials $U_i(M_i) = V_i(M_i) + 2cM_i^2$ have here been shifted by a quadratic term). We will consider the resolvent of the first matrix M_0†

$$\omega(z) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z - M_0}\right\rangle.$$

The loop equations are obtained by performing the changes of variables

$$M_i \longrightarrow M_i + \epsilon\frac{1}{z - M_0}M_p^{n_p}M_{p-1}^{n_{p-1}}\ldots M_{i+1}^{n_{i+1}}$$

† In fact, one can easily obtain a closed algebraic equation in the large N limit only for the resolvent of the first ($i = 0$) and last ($i = p$) matrices.

and by multiple recurrences on the n_i's. The proof is quite tedious but not difficult. We will first need some notation. We define the function

$$z_0(z) = z \qquad z_1(z) = \frac{1}{2c}(U_0'(z) - \omega(z))$$

and

$$z_{i+1}(z) = \frac{1}{2c}U_i'(z_i(z)) - z_{i-1}(z) \qquad \text{for } 0 < i < p. \tag{D.1}$$

Each function $z_i(z)$ is a polynomial in z and $\omega(z)$.

The algebraic equation satisfied by the resolvent $\omega(z)$ is then

$$\left(\frac{1}{2c}U_0'(z_0(z)) - z_1(z)\right)\left(\frac{1}{2c}U_p'(z_p(z)) - z_{p-1}(z)\right) = \frac{1}{2c}P(z_0(z), \ldots, z_p(z)) \tag{D.2}$$

where the right-hand side $P(\zeta_0, \ldots, \zeta_p)$ is a polynomial of $p+1$ variables (now considered as independent),

$$P(\zeta_0, \ldots, \zeta_p) = \frac{1}{N}\langle \operatorname{tr} F_{p+1}\rangle$$

and the F_n's are defined as follows:

$$F_0 = 1 \qquad F_1 = \frac{1}{2c}\frac{U_0'(\zeta_0) - U_0'(M_0)}{\zeta_0 - M_0} \qquad F_{n+1} = \frac{1}{2c}\frac{U_n'(\zeta_n) - U_n'(M_n)}{\zeta_n - M_n}F_n - F_{n-1}.$$

Note that (D.2) looks symmetric under the exchange of the two extremities of the chain $i \to p - i$.

As in the one-matrix case, most of the coefficients of the polynomial P are unknown; they have to be fixed by some analytical considerations about $\omega(z)$, for instance impose that $\omega(z)$ has only one cut. This one-cut assumption should allow one, in principle, to determine all the unknown coefficients of P.

We have made exactly the same assumption when we have replaced the operator $\hat{x}$ by a number x in the large N limit (cf section 1.3), and we observe that the functions $\lambda_i(x)$ give a parametrization of equation (D.1). In fact, if $z = \lambda_0(x)$ then each $z_i(z)$ is nothing but $\lambda_i(x)$, and the loop equation (D.2) is nothing but the product of (1.16) and (1.17).

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